Representation of functions in Hyland-Ong games

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Introduction

Game semantics provide precise models of various programming languages. Around 1994, Abramsky et al. [AMJ94] and Hyland and Ong [HO00] proposed two constructions of a full abstract model of the purely functional language PCF. We focus on the model of Hyland and Ong, which has been adapted to handle control operators [Lai97] and references [AHM98]. In these games, plays are traces of interaction between a program (player P) and an environment (opponent O). A program is interpreted by a strategy for P which represents the interactions it can have with any environment. Control operators and references are modeled by relaxing constraints on the strategies corresponding to pure functional programming. Different programming primitives can thus be modeled in a common framework parametrized by the class of strategies considered.

This work studies the representation of set-theoretic functions on infinite sequences by strategies of Hyland-Ong games, and conversely, wether these strategies represent some set-theoretic functions on infinite sequences.The main motivation is the possible definition of (modified) realizability models (see e.g. [Tro98]) based on game semantics.

We define a notion of representation of functions on infinite sequences and investigate the following questions: wether representable functions are continuous; wether continuous functions are representable; and wether strategies of a given class do represent a function.

This report is organized as follows. The first part defines existing notions of topology and Hyland-Ong games. In a second part we define our notion of representation. The third part is devoted to the proof of equivalence between continuous and representable functions. In the fourth part we give some sufficient conditions for a strategy to represent a function.

1 Preliminaries

1.1 Topology

If α is a sequence in X^ω (the set of infinite sequences on set X), then α_i denotes the i^{th} term of α. We give the set of sequences of a set X the product topology of the discrete topology on X. A basis of open sets of X^ω is then the set of $O_{x_0 \dots x_n}$ where $x_0 \dots x_n$ is a finite sequence on X, defined by:

$$O_{x_0 \dots x_n} = \{\alpha \in X^\omega | \alpha_0 = x_0 \dots \alpha_n = x_n\}$$

and a prebasis is the set of:

$$\{\alpha \in X^\omega | \alpha_n = x\}$$

where $x \in X$ and $n \in \omega$.

Therefore, if $X = \{0, 1\}$, we get the Cantor space, and if $X = \omega$, we get the Baire space (see [Kec95]). In this context, a function f from X^ω to Y^ω is continuous if:

$$\forall \alpha \in X^\omega, \forall m \in \omega, \exists n \in \omega, \forall \beta \in X^\omega, \beta_0 \dots \beta_n = \alpha_0 \dots \alpha_n \Rightarrow f(\beta)_m = f(\alpha)_m$$

The idea is that in order to have a token of information on the output (the value of $f(\alpha)_m$), we only need to know a finite amount of information on the input.

1.2 Hyland-Ong games

In Hyland-Ong games, programs are interpreted by strategies and plays (sequences of moves) represent execution traces. We mainly use the definitions of Harmer's PhD thesis [Har99].

1.2.1 Arenas, plays and strategies

Plays occur in arenas which are sets of moves together with an enabling relation.

Definition 1 (Arenas)**.** *An arena A is a set of moves $\mathcal{M}_A$ together with a binary relation $\vdash_A \subseteq \mathcal{M}_A \times \mathcal{M}_A$, called enabling, which induces a forest on $\mathcal{M}_A$.*

There are two players: player (P) and opponent (O), who play moves alternatively.

Definition 2 (Polarity)**.** *To each move m of an arena A, we associate a polarity in $\{O; P\}$, depending on the parity of the depth of m into the forest induced by $\vdash_A$: if m is at even depth (which is the case for the roots), then m is given the polarity O, otherwise m is given the polarity P.*

The O-moves are the moves made by the environment, whereas the P-moves are those made by the program. The moves will be often be denoted as $a_X^{\mathcal{P}}$ where a belongs to X and $\mathcal{P}$ is the polarity of the move (which depends on the arena). A move which is a leaf is called an answer, and all other moves are called questions (in other settings, a leaf can be a question and an answer is not necessarily a leave, see for instance [AHM98]).

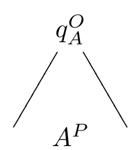
We are only interested in arenas which are inductively built from flat arenas and the arrow constructor. These arenas will always be trees.

Definition 3.

- *If A is set, the corresponding flat arena is defined by:*

$$\mathcal{M}_A = \{q_A^O\} \cup \{a^P | a \in A\} \quad \textit{and} \quad \forall a \in A,\ q_A^O \vdash_A a^P$$

which can be depicted as:

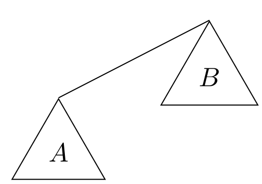


- *If A and B are arenas, the arena $A \to B$ is defined by:*

$$\mathcal{M}_{A\to B} = \mathcal{M}_A \uplus \mathcal{M}_B$$

$$m_1 \vdash_{A\to B} m_2 \quad \Longleftrightarrow \quad \begin{cases} m_1 \in \mathcal{M}_A \\ m_2 \in \mathcal{M}_A \\ m_1 \vdash_A m_2 \end{cases} \quad \textit{or} \begin{cases} m_1 \in \mathcal{M}_B \\ m_2 \in \mathcal{M}_B \\ m_1 \vdash_B m_2 \end{cases} \quad \textit{or} \begin{cases} m_1 \in \mathcal{M}_B \\ m_2 \in \mathcal{M}_A \\ \forall m_1' \in \mathcal{M}_B, m_1' \nvdash_B m_1 \\ \forall m_2' \in \mathcal{M}_B, m_2' \nvdash_B m_2 \end{cases}$$

Schematically, if A and B are arenas which are trees, the arena $A \to B$ is:



Remark that the polarity of a move in A is the opposite of its polarity in $A \to B$.

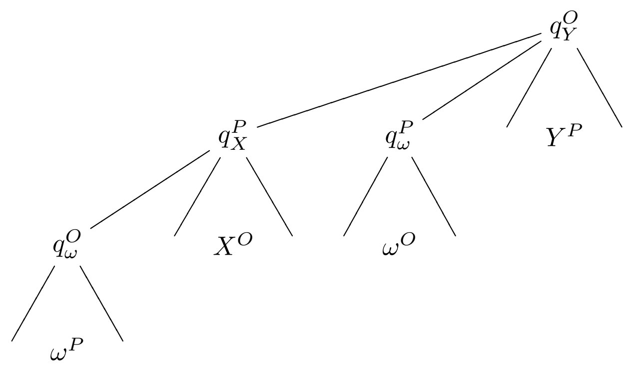


Figure 1: the arena $(\omega \to X) \to (\omega \to Y)$

We are mainly interested in working on maps from sequences to sequences, the corresponding arena being described in Figure 1. The moves of this arena are the following: q_Y^O, q_X^P, q_ω^P, q_ω^O are moves, and Y^P, X^O, ω^O, ω^P are sets of moves respectively isomorphic to Y, X, ω, ω. The moves which are questions are denoted q, with the concerned set in subscript and the polarity in superscript.

Plays are represented by pointed sequences, which are sequences of moves, some of them having a pointer, which points to a preceding move.

Definition 4 (Pointed sequence)**.** *A pointed sequence of moves in the arena A is a word $w \in \mathcal{M}_A^*$ together with a pointer function $f : \{0; \dots ; |w| - 1\} \to \{-1; \dots ; |w| - 1\}$ such that $\forall i, f(i) < i$.*

This means that if (w, f) is a pointed sequence, the move w_i points to $w_{f(i)}$ if $f(i) \neq -1$, and w_i doesn't have a pointer if $f(i) = -1$. If w_i points to w_j, we say that w_j enables w_i and that w_i is justified by w_j. From now on we do not write explicitely the pointer function.

Definition 5 (Pointed prefix)**.** *If w is a pointed sequence, we define for $k \in \{-1; \dots ; |w| - 1\}$ the prefix of w ending with w_k (denoted $w_{|k}$) which keeps the same pointers.*

It is then easy to see that $w_{|k}$ is a pointed sequence. A play is a pointed sequence on which we impose some constraints:

Definition 6 (Play)**.** *A play on the arena A is a pointed sequence of moves w which is alternating (the polarities of moves in w alternate), and which satisfies:*

$$\forall i, \begin{cases} \text{if } w_i \text{ is justified by } w_j\text{, then } w_j \vdash_A w_i \\ \text{if } w_i \text{ is not justified, then } \forall m \in \mathcal{M}_A, m \nvdash_A w_i \text{ (i.e.: } w_i \text{ is a root of } A\text{)} \end{cases}$$

If w is a play, then a move m with its pointer is be said legal if wm is still a play. By definition of a pointed sequence, if w is a non-empty play, then w_0 is not justified, which means that w_0 is a root of the arena, and therefore has polarity O, so by alternation, we get for any play w:

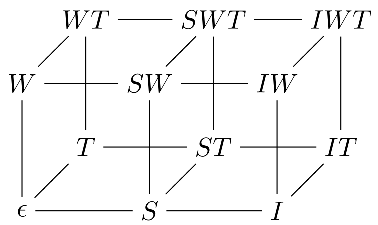
$$\forall i, w_{2i} \text{ has polarity } O\text{, and } w_{2i+1} \text{ has polarity } P$$

We now define strategies.

Definition 7 (Strategy)**.** *A strategy σ on the arena A is a non-empty set of even-length plays, which is closed under even-length prefixes and which is deterministic: if w and w' are plays in σ such that:*

$$|w'| = |w| = n \wedge w'_{|n-2} = w_{|n-2}$$

then we have $w'_{n-1} = w_{n-1}$, which means $w' = w$.



$S =$ single threaded
$I =$ innocent
$W =$ well bracketed
$T =$ total
$\epsilon =$ all strategies

Figure 2: The class of strategies parallelepiped

If we are given two strategies σ and τ respectively on the arenas $A \to B$ and $B \to C$, then we can compose them to obtain the strategy $\sigma;\tau$ on the arena $A \to C$. The polarities of the moves in B being opposite in the arenas $A \to B$ and $B \to C$, we can copy each P-move of σ in B as a O-move in $B \to C$ and each P-move of τ in B as a O-move in $A \to B$. The identity on arena $A \to A$ uses the same trick: a O-move in one of the A is a P-move in the other and conversely a P-move in one of the A is a O-move in the other. The identity is then a simple copycat strategy. This gives us a category which objects are arenas and morphisms are strategies.

We use the application of strategies, which is slightly simpler than the composition: if σ is a strategy on $A \to B$ and if τ is a strategy on A, τ can be seen as a strategy on $\top \to A$, so the composition $\tau;\sigma$ is a strategy on $\top \to B$, which can be seen as a strategy on B. Then $\sigma(\tau)$ denotes the application of τ to σ, which is a strategy on B. Application can also be defined more shortly as:

$$\sigma(\tau) = \{u_{|B} | u_{|B} \text{ alternating } \wedge u \in \sigma \wedge u_{|A} \in \tau\}$$

1.2.2 Constraining strategies

A important parameter of strategies is to which extent they have access to the history of a play. In the above definition, strategies have access to the whole history. However, precise interpretations of programming languages require some restriction on the available history, such as single threadness and innocence. It is also possible to restrict the control flow, for instance by imposing a bracketing condition. Innocence together with well bracketing corresponds to purely functional languages (see [HO00]). Relaxing well bracketing allows to model control operators (see [Lai97]), and relaxing innocence to single threadness allows to model references (see [AHM98]). The corresponding classes of strategies are depicted in the parallelepiped of Figure 2. We now give the definitions of these restrictions.

We first define the notion of single threaded strategy. If w is a non-empty play, then by following the pointers from the last move, we eventually find a move which is not justified. This move is called the hereditary justifier of w and is denoted $HJ(w)$. The thread $\lceil w \rceil$ of a play w is the pointed subsequence containing the moves of the play which have the same hereditary justifier as that of the last move of the play. More precisely:

Definition 8 (Thread)**.** *If w is empty, then $\lceil w \rceil = \lceil \epsilon \rceil = \epsilon$. If w is a non-empty play, let write:*

$$I = \{0 \leq i < |w|, HJ(w_{|i}) = HJ(w)\} = \{i_0; \ldots; i_p\}$$

with $i_0 < i_1 < \cdots < i_p$, then $\lceil w \rceil$ is $w_{i_0} \ldots w_{i_p}$ with pointers inherited from w.

A play is said well threaded if each P-move of w has the same hereditary justifier as the preceding O-move. A well threaded play is a play in which P plays in the same thread as the one of the last O-move. This is equivalent to say that each P-move points to the current thread. In a well threaded play, only O can switch thread. We have the property that the thread of a well threaded play is itself a play. Indeed, we just have to verify that it is alternating. This is the case because after a O-move (resp. before a P-move) of the thread comes a P-move (resp. a O-move) (by alternation of the initial play) which is in the same thread (by well threadedness). Well threaded plays are defined as follows:

Definition 9 (Well threaded play)**.** *A play w is well threaded if:*

$$\forall i \geq 0, 2i+1 < |w| \Rightarrow HJ(w_{|2i+1}) = HJ(w_{|2i})$$

A single threaded strategy is such that if after a O-move, the current thread is answered in some play of the strategy, then it is answered in the current play, with the same answer. This means that the strategy cannot benefit from the informations of the other threads of the current play. The precise definition is:

Definition 10 (Single threaded strategies)**.** *A strategy σ is single threaded if all its plays are well threaded and if for two well threaded plays w, w' with $|w| = n, |w'| = n'$ satisfying:*

$$w \in \sigma, w'_{|n'-3} \in \sigma, \lceil w \rceil = \lceil w' \rceil$$

we have $w' \in \sigma$.

We now define the notion of innocent strategy. The view $\ulcorner w \urcorner$ of a play w is obtained by going backwards from the end of w, jumping over P-moves and following all the pointers from O-moves (and therefore stopping on a root). The formal definition is:

Definition 11 (View)**.** *The view is defined by:*

1. $\ulcorner \epsilon \urcorner = \epsilon$

2. $\ulcorner wa^P \urcorner = \ulcorner w \urcorner a^P$ *with a^P pointing to the same move as before if it is in $\ulcorner w \urcorner$, otherwise it has no pointer*

3. $\ulcorner wa^P v b^O \urcorner = \ulcorner wa^P \urcorner b^O$ *if b^O points to a^P*

A given play satisfies visibility if for every prefix wa^P of it, a^P points to a move in $\ulcorner w \urcorner$. This means that the justifier of any P-move is not jumped over by some O-move pointer. If a play satisfies visibility, then its view is itself a play, in which each O-move points to the preceding move. It is not difficult to see that a play which satisfies visibility is in particular well threaded, and that in this case, the view is a pointed subsequence of the thread.

A strategy is said innocent if it satisfies: if after a O-move, the current view is answered in some play of the strategy, then it is answered in the current play, with the same answer. This means that the strategy cannot benefit from the informations of the other threads nor the moves between a O-move and its justifier. The class of innocent strategies is formally defined as:

Definition 12 (Innocent strategies)**.** *A strategy σ is innocent if all its plays satisfy visibility and if for two well threaded plays w, w' with $|w| = n, |w'| = n'$ satisfying:*

$$w \in \sigma, w'_{|n'-3} \in \sigma, \ulcorner w \urcorner = \ulcorner w' \urcorner$$

we have $w' \in \sigma$.

Finally we define the notion of well bracketed strategy. Well bracketed plays are such that each answer is an answer to the most recent unanswered question. This is a constraint on the control flow. Formally:

Definition 13 (Well bracketed play)**.** *A play w is well bracketed if for each $0 \leq i < |w|$, if w_i is an answer pointing to the question w_j, then:*

- *for all w_k between w_j and w_i, if w_k is a question, then there is a move between w_k and w_i which is an answer and which points to w_k*

- *for all w_k between w_j and w_i, if w_k is an answer, then it does not point to w_j*

A well bracketed strategy is a strategy which preserves well bracketing:

Definition 14 (Well bracketed strategies)**.** *A strategy σ is well bracketed if for any $w \in \sigma$ with $|w| = n$, if $w_{|n-2}$ is well bracketed then so is w.*

A total strategy is a strategy which can always answer to a legal O-move:

Definition 15 (Total strategies)**.** *A strategy σ is total if for any play w with $|w| = n$, if $w_{|n-2} \in \sigma$, then there exists $w' \in \sigma$ such that $|w'| = n + 1$ and $w'_{|n-1} = w$.*

From now on, $\mathcal{C}$ will denote a class of strategies which can be any node in the parallelepiped of Figure 2.

If we restrict the category of arenas and strategies to the particular cases of single threaded/innocent/well bracketed strategies, we obtain a sub-category. Indeed, it is true that all the classes of strategies on the front face of the parallelepiped of Figure 2 are stable by composition and contain the copycat strategy (see for instance [Har99]). Innocence and well bracketing are preserved by application.

2 Representation of functions

In this section we introduce and define the main concept we are interested in: what it means for a strategy σ on the arena $(\omega \to X) \to (\omega \to Y)$ to represent a function f from X^ω to Y^ω. Representation is parameterized by a class $\mathcal{C}$ of strategies and denoted $\sigma \Vdash_{\mathcal{C}} f$.

Our definition is inpired by realizability and proceeds by induction on the arenas which are inductively built according to Definition 3. To each such arena we associate a set interpretation as follows:

- The set interpretation of a flat arena A is A^P (the set of its leaves).
- The set interpretation of the arrow arena $A \to B$ is the set of functions from the set interpretation of A to the set interpretation of B.

Hence the set interpretation of the arena $(\omega \to X) \to (\omega \to Y)$ is the set of functions from X^ω to Y^ω. We write $a \in A$ if a belongs to the set interpretation of the arena A.

Definition 16 (Representation $\Vdash_{\mathcal{C}}$)**.** *Let $\mathcal{C}$ be a class of strategies.*

Let A be an arena inductively defined according to Definition 3. Representation $\sigma \Vdash_{\mathcal{C}} a$ of an element $a \in A$ of the set interpretation of the arena A by strategy σ on the arena A w.r.t. $\mathcal{C}$ is defined by induction on the construction of A:

- *If A is a flat arena and $a \in A$, we have $\sigma \Vdash_{\mathcal{C}} a$ iff σ is in $\mathcal{C}$ and contains the play:*

$$\begin{array}{c} A \\ q \\ a \end{array}$$

- *If A is the arena $B \to C$, and $f : B \to C$, then $\sigma \Vdash_{\mathcal{C}} f$ iff σ is in $\mathcal{C}$ and forall $b \in B$ and $\tau \Vdash_{\mathcal{C}} b$, we have $\sigma(\tau) \Vdash f(b)$.*

An element is said to be representable if it is represented by a strategy.

Remark 1. *The choice for the representation of flat arenas can be discussed. Indeed, we ask for a representant to answer the represented object at the first time. Another choice would have been to ask the representant to always answer the represented object, which is a more rigid definition and allows to remove the single threadedness condition in some propositions. Yet another choice would have been to ask for the strategy to answer at least once the represented object, in which case a strategy can represent countably many objects. This choice is still to be investigated.*

We define strategies representing arbitrary integers and sequences.

Definition 17. • *If $m \in \omega$, define τ_m as the innocent strategy which views are even-length subplays of:*

$$\begin{array}{c} \omega \\ q_\omega^O \\ m^P \end{array}$$

i.e.: $\tau_m = (q_\omega^O m^P)^*$.

- *If $\alpha \in X^\omega$, define τ_α as the innocent strategy which views are even-length subplays of:*

$$\begin{array}{ccc} \omega & \to & X \\ & & q_X^O \\ q_\omega^P & & \\ n^O & & \\ & & \alpha_n^P \end{array}$$

These strategies are in some sense "canonical", and it is easy to see that they have good properties:

Proposition 1. • *τ_m and τ_α are innocent.*

- *τ_m and τ_α are well bracketed.*
- *For any $\mathcal{C}$, $\tau_m \Vdash_\mathcal{C} m$ and $\tau_\alpha \Vdash_\mathcal{C} \alpha$*

Proof. We only prove the third point:
ω being a flat arena, we immediately have $\tau_m \Vdash_\mathcal{C} m$.
Let now $m \in \omega$ and $\tau \Vdash_\mathcal{C} m$, i.e. τ contains the play:

$$\begin{array}{c} \omega \\ q_\omega^O \\ m^P \end{array}$$

τ_α contains the following play w:

$$\begin{array}{ccc} \omega & \to & X \\ & & q_X^O \\ q_\omega^P & & \\ m^O & & \\ & & \alpha_m^P \end{array}$$

Moreover, $w_{|\omega} \in \tau$ and $w_{|X}$ is the play:

$$\begin{array}{c} X \\ q_X^O \\ \alpha_m^P \end{array}$$

So we have $\tau_\alpha(\tau) \Vdash_\mathcal{C} \alpha_m$, and so $\tau_\alpha \Vdash_\mathcal{C} \alpha$. □

3 Representability and continuity

In this section we study the connection between representability and continuity. We show that:

- For any $\mathcal{C}$, a $\mathcal{C}$-representable function is continuous.
- Any continuous function is $\mathcal{C}$-representable if $\mathcal{C}$ contains only single threaded strategies.

We also give a counter-example which proves that the single threadness hypothesis is necessary for the second point.

3.1 Every representable function is continuous

It is quite simple to see that a map from sequences to sequences which is representable is continuous. We've seen that a continuous function is a function for which a finite information on the output is determined by a finite information on the input. On the other hand an interaction is a finite sequence of moves, so only a finite number of terms of the input appear in it.

Theorem 1. *For any $\mathcal{C}$, if $f : (\omega \to X) \to (\omega \to Y)$ is $\mathcal{C}$-representable, then f is continuous.*

Proof. Let $\sigma \Vdash_{\mathcal{C}} f$, let $\alpha \in X^\omega$, and let $m \in \omega$. We have $\tau_\alpha \Vdash_{\mathcal{C}} \alpha$ and $\tau_m \Vdash_{\mathcal{C}} m$ by Proposition 1, so $\sigma(\tau_\alpha)(\tau_m) \Vdash_{\mathcal{C}} f(\alpha)_m$, and so $q_Y^O f(\alpha)_m^P \in \sigma(\tau_\alpha)(\tau_m)$. Then by definition of application:

$$\exists u \in \sigma(\tau_\alpha), u_{|\omega} \in \tau_m \wedge u_{|Y} = q_Y^O f(\alpha)_m^P$$

$$\exists v \in \sigma, v_{|\omega \to X} \in \tau_\alpha \wedge v_{|\omega \to Y} = u$$

Then since v is a finite word, there is a N such that for each occurence n^P in v, we have $n \leq N$. Let now $\beta \in X^\omega$ such that $\forall n \leq N, \beta_n = \alpha_n$. Suppose $v_{|\omega \to X} \notin \tau_\beta$. Then there is a prefix w of $v_{|\omega \to X}$ such that $\ulcorner w \urcorner \in \tau_\alpha \setminus \tau_\beta$. This means that $\ulcorner w \urcorner$ is a play of the form:

$$\begin{array}{ccc}
\omega & \to & X \\
 & & q_X^O \\
q_\omega^P & & \\
n^O & & \\
 & & \alpha_n^P
\end{array}$$

for some n such that $\alpha_n \neq \beta_n$. But then we have $n > N$ and $n^P \in v$, which is impossible. So $v_{|\omega \to X} \in \tau_\beta$ and $u \in \sigma(\tau_\beta)$. It follows that $\sigma(\tau_\beta)(\tau_m)$ contains the play:

$$\begin{array}{c}
Y \\
q_Y^O \\
f(\alpha)_m^P
\end{array}$$

Therefore we have $\sigma(\tau_\beta)(\tau_m) \Vdash_{\mathcal{C}} f(\alpha)_m$, but because $\sigma \Vdash_{\mathcal{C}} f$, $\tau_\beta \Vdash_{\mathcal{C}} \beta$, and $\tau_m \Vdash_{\mathcal{C}} m$, we also have $\sigma(\tau_\beta)(\tau_m) \Vdash_{\mathcal{C}} f(\beta)_m$, so $f(\alpha)_m = f(\beta)_m$ by unicity of the represented object. We finally proved that:

$$\forall \alpha \in \omega \to X, \forall m \in \omega, \exists N \in \omega, \forall \beta \in \omega \to X, (\forall n \leq N, \beta_n = \alpha_n) \Rightarrow f(\beta)_m = f(\alpha)_m$$

which means that f is continuous. $\square$

3.2 Representation of continuous functions

For the representation of a continuous function f, we define an innocent well bracketed strategy σ_f which represents f, so σ_f must behave well with any representant of a sequence. The first thing we prove is that the representants of α are in a sense not very far from τ_α:

Proposition 2. *For any $\mathcal{C}$, if $\rho \Vdash_{\mathcal{C}} \alpha \in X^\omega$, then for all m, there is a n such that ρ contains the play:*

$$\begin{array}{ccc}
\omega & & \overbrace{q_\omega^P \; m^O \; \cdots \; q_\omega^P \; m^O}^{n\times} \\
\downarrow & & \\
X & q_X^O & \qquad\qquad\qquad \alpha_m^P
\end{array}$$

Proof. The strategy τ_m is in $\mathcal{C}$ and we have $\tau_m \Vdash_{\mathcal{C}} m$ by Proposition 1, so:

$$
\begin{aligned}
\rho \Vdash_{\mathcal{C}} \alpha &\Longrightarrow \forall m, \rho(\tau_m) \Vdash_{\mathcal{C}} \alpha_m \\
&\Longrightarrow \forall m, q_X^O \alpha_m^P \in \rho(\tau_m) \\
&\Longrightarrow \forall m, \exists u \in \rho, u_{|\omega} \in \tau_m \wedge u_{|X} = q_X^O \alpha_m^P \\
&\Longrightarrow \forall m, \exists n, \rho \text{ contains: }
\begin{array}{cccc}
\omega & & \overbrace{q_\omega^P \; m^O \; \ldots \; q_\omega^P \; m^O}^{n\times} & \\
\downarrow & & & \\
X & q_X^O & & \alpha_m^P
\end{array}
\end{aligned}
$$

∎

We now give the definition of the innocent well bracketed σ_f which will represent f:

Definition 18. *If $f : (\omega \to X) \to (\omega \to Y)$ is continuous, we define the total innocent well bracketed strategy σ_f by its views:*
If $m \in \omega$ and $\alpha \in X^\omega$, let k be the modulus of continuity of f at α, m:

$$k = \min\{i | \forall \beta \in X^\omega, \beta_0 \ldots \beta_i = \alpha_0 \ldots \alpha_i \Rightarrow f(\beta)_m = f(\alpha)_m\}$$

which is finite by continuity of f, then σ_f contains the even-length prefixes of:

$$
\begin{array}{ccccccc}
(\omega & \to & X) & \to & (\omega & \to & Y) \\
 & & & & & & q_Y^O \\
 & & & & q_\omega^P & & \\
 & & & & m^O & & \\
 & & q_X^P & & & & \\
 & & \alpha_0^O & & & & \\
 & & \vdots & & & & \\
 & & q_X^P & & & & \\
q_\omega^O & & & & & & \\
i^P & & & & & &
\end{array}
\qquad \textit{and} \qquad
\begin{array}{ccccccc}
(\omega & \to & X) & \to & (\omega & \to & Y) \\
 & & & & & & q_Y^O \\
 & & & & q_\omega^P & & \\
 & & & & m^O & & \\
 & & q_X^P & & & & \\
 & & \alpha_0^O & & & & \\
 & & \vdots & & & & \\
 & & q_X^P & & & & \\
 & & \alpha_k^O & & & & \\
 & & & & & & f(\alpha)_m^P
\end{array}
$$

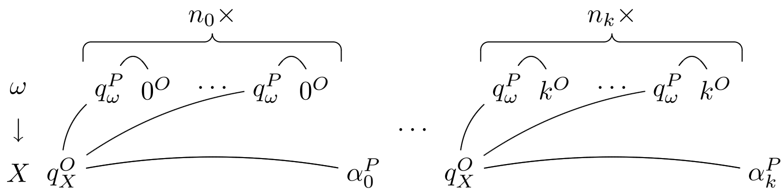
Finally, we prove by innocence of σ_f that it indeed represents f:

Theorem 2. *If $\mathcal{C}$ contains only single threaded strategies, if $f : (\omega \to X) \to (\omega \to Y)$ is continuous, then $\sigma_f \Vdash_{\mathcal{C}} f$.*

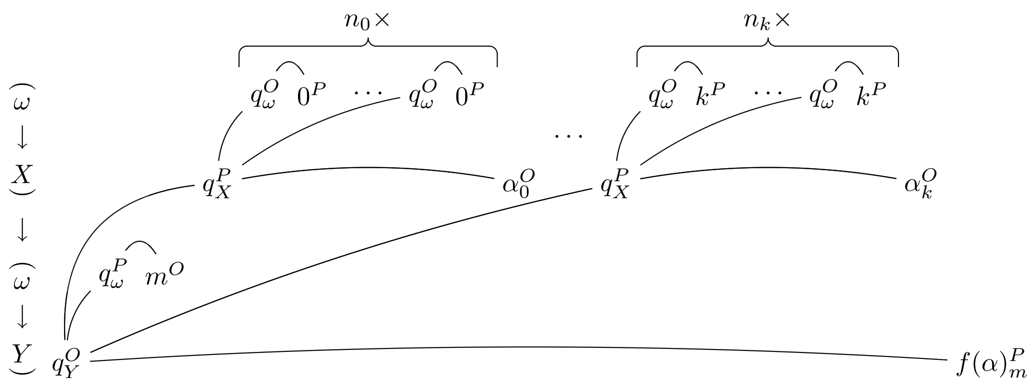
Proof. Let $\alpha \in X^\omega$, $m \in \omega$, and k the modulus of continuity defined as above. Let $\tau \Vdash_{\mathcal{C}} m$ and $\rho \Vdash_{\mathcal{C}} \alpha$. From Proposition 2, let $n_0, \ldots, n_k$ be such that ρ contains:

$$
\begin{array}{cccc}
\omega & & \overbrace{q_\omega^P \; i^O \; \ldots \; q_\omega^P \; i^O}^{n_i\times} & \\
\downarrow & & & \\
X & q_X^O & & \alpha_i^P
\end{array}
$$

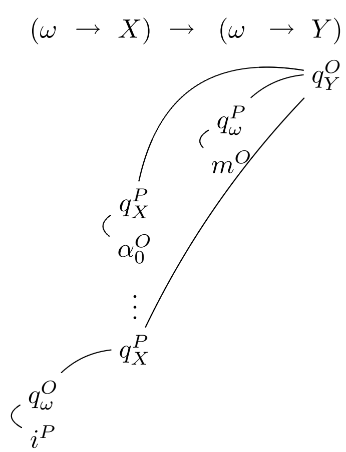
ρ is single threaded by hypothesis on $\mathcal{C}$, so we know that it contains:



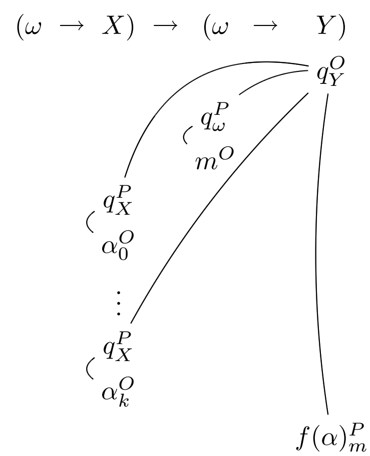
We show that σ_f contains the following play w:



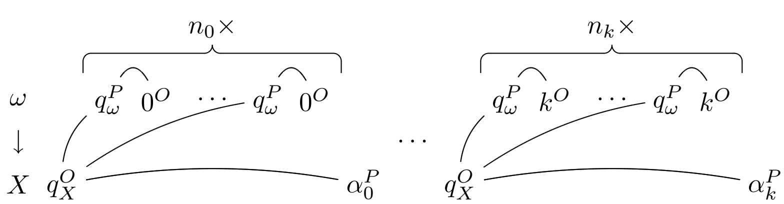
Indeed the view of the even-length prefixes of w are all prefixes of:



and



which are, by definition, in σ_f. Moreover, $w_{|\omega\to X}$ is the following play:



which is in ρ, and we have $w_{|\omega} = q_\omega^O m^P \in \tau$ since $\tau \Vdash_{\mathcal{C}} m$. Therefore, we have $w_{|Y} = q_Y^O f(\alpha)_m^P \in \sigma_f(\rho)(\tau)$, so $\sigma_f(\rho)(\tau) \Vdash_{\mathcal{C}} f(\alpha)_m$. Finally, we get that $\sigma_f \Vdash_{\mathcal{C}} f$. $\square$

We now show that the hypothesis on $\mathcal{C}$ is necessary for σ_f to represent f. For this we first give some pathological non single threaded representants of sequences which only responds to the first answer:

Proposition 3. *Let τ_α^{pat} be the strategy containing the even-length prefixes of the plays:*

$$
\begin{array}{ccc}
\omega & \to & X \\
 & & q_X^O \\
q_\omega^P & & \\
n^O & & \\
 & & \alpha_n^P \\
 & any^O & \\
q_\omega^P & & \\
 & \vdots & \\
 & any^O & \\
q_\omega^P & &
\end{array}
\qquad \textit{and} \qquad
\begin{array}{ccc}
\omega & \to & X \\
 & & q_X^O \\
q_\omega^P & & \\
 & & q_X^O \\
q_\omega^P & & \\
 & any^O & \\
q_\omega^P & & \\
 & \vdots & \\
 & any^O & \\
q_\omega^P & &
\end{array}
$$

then we have:

$$\tau_\alpha^{pat} \Vdash_{\mathcal{C}} \alpha$$

if $\mathcal{C} \in \{\epsilon; Wb; T; WbT\}$.

Proof. It is easy to see that τ_α^{pat} is total and well bracketed. Let $n \in \omega$ and $\tau \Vdash_{\mathcal{C}} n$. Then $q_\omega^O n^P \in \tau$. Since

$$
w = \begin{array}{ccc}
\omega & \to & X \\
 & & q_X^O \\
q_\omega^P & & \\
n^O & & \\
 & & \alpha_n^P
\end{array}
\in \tau_\alpha^{pat} \wedge w_{|\omega} = q_\omega^O n^P \in \tau \wedge w_{|X} = q_X^O \alpha_n^P
$$

we get that $q_X^O \alpha_n^P \in \tau_\alpha^{pat}(\tau)$, so $\tau_\alpha^{pat}(\tau) \Vdash_{\mathcal{C}} \alpha_n$, and so $\tau_\alpha^{pat} \Vdash_{\mathcal{C}} \alpha$. $\square$

Now we define a continuous function which, on the input 0^ω, needs to know strictly more than one term in order to determine the ouput:

Proposition 4. *The function:*

$$
\begin{aligned}
f : \{0;1\}^\omega &\longrightarrow \{0;1\}^\omega \\
00\alpha &\longmapsto 0^\omega \\
01\alpha &\longmapsto 1^\omega \\
10\alpha &\longmapsto 1^\omega \\
11\alpha &\longmapsto 1^\omega
\end{aligned}
$$

is continuous, but non representable for $\mathcal{C} \in \{\epsilon; W; T; WT\}$.

Proof. f is obviously continuous. Suppose σ is such that $\sigma \Vdash_{\mathcal{C}} f$ with $\mathcal{C} \in \{\epsilon; Wb; T; WbT\}$, then for all α we have:

$$\sigma(\tau_\alpha^{pat})(\tau_0) \Vdash_{\mathcal{C}} f(\alpha)_0$$

For $\alpha = 0^\omega$, we have $f(\alpha)_0 = 0$, so $\sigma(\tau^{pat}_\alpha)(\tau_0) \Vdash_{\mathcal{C}} 0$, and so $q^O_Y 0^P \in \sigma(\tau^{pat}_\alpha)(\tau_0)$. Then there is a $w \in \sigma$ such that $w_{|\omega} \in \tau_0$, $w_{|\omega \to X} \in \tau^{pat}_\alpha$, and $w_{|Y} = q^O_Y 0^P$. We then define a $\beta \in \{0;1\}^\omega$ such that $f(\beta)_0 \neq f(\alpha)_0$, depending on wether σ asks for the first term of α, i.e. wether if

$$
\begin{array}{ccc}
\omega & \to & X \\
 & & q^O_X \\
q^P_\omega & & \\
0^O & & \\
 & & 0^P
\end{array}
$$

is a prefix of $w_{|\omega \to X}$:

- if so, then $\beta = 01^\omega$ (β has the same first term as α),
- if not, then $\beta = 10^\omega$ (β has the same other terms as α).

We still have $w_{|\omega \to X} \in \tau^{pat}_\beta$, so:

$$\sigma(\tau^{pat}_\beta)(\tau_0) \Vdash_{\mathcal{C}} 0$$

which is impossible, since $f(\beta)_0 = 1$, because one strategy cannot represent two different integers. $\square$

4 From innocent well bracketed strategies to functions

Here we prove that an innocent well bracketed strategy which is hereditarily total (to be defined) represents a function in a class of strategies which contains only innocent strategies. The principle is for the candidate representant σ to:

- make sure that σ gives an answer when playing against τ_α and τ_m (this comes from hereditary totality)
- use this answer to define the function
- given τ'_α and τ'_m two other representants of α and m, modify the interaction between σ, τ_α and τ_m in order to obtain the good play in $\sigma(\tau'_\alpha)(\tau'_m)$

For the third point, we describe the interactions of σ with τ_α and τ_m we are interested in. We show that these are of the form $q^O_Y v y^P$ with v generated by the grammar:

$$\mathbf{Q} \ ::= \ \epsilon \ \mid \ \mathbf{Q} \ q^O_X \ q^P_\omega \ \mathbf{Q} \ i^O \ \alpha^P_i \ \mid \ \mathbf{Q} \ q^P_\omega \ m^O$$

where in fact the pointers are unnecessary (c.f.: Remark 2). These interaction sequences are then modified into interaction sequences of σ with τ'_α and τ'_m.

The proof of the fact that the play obtained after the modification indeed corresponds to an interaction in $\sigma(\tau'_\alpha)(\tau'_m)$ uses the innocence of τ'_α, and this is why we need to restrict to classes containing only innocent strategies.

The hypothesis of well bracketing of σ is necessary, as shown in Proposition 9.

4.1 Heriditarily total strategies

For the first part of the proof, the strategy has to give an answer to any pair of representants of a sequence and an integer. For that, we would want to restrict ourselves to total strategies, but the composition of total strategies is not necessarily total: an infinite interaction can appear in the arena B when composing strategies in the arenas $A \to B$ and $B \to C$. In the context of innocent strategies and finite arenas, we usually restrict to the case of finite strategies, that is, those for which the set of views of the plays it contains is finite. In that case, innocent total finite strategies do compose. However, since our sets can be infinite, the notion of finiteness of strategies doesn't give sense anymore, but total strategies still do not compose, so we define the class of hereditarily total strategies, which is stable by application.

Definition 19 (Hereditarily total strategies)**.** *We define the set $HT_{\mathcal{C}}(A)$ of hereditarily total strategies for class $\mathcal{C}$ on arena A inductively built according to Definition 3 as follows:*

- *If A is a flat arena, a strategy on this arena is in $HT_{\mathcal{C}}(A)$ if it is total and in $\mathcal{C}$.*
- *A strategy σ is in $HT_{\mathcal{C}}(A \to B)$ if for any strategy τ in $HT_{\mathcal{C}}(A)$, $\sigma(\tau)$ is in $HT_{\mathcal{C}}(B)$.*

The canonical representants τ_α and τ_m are indeed hereditarily total:

Proposition 5. *For any $\mathcal{C}$, $\tau_m \in HT_{\mathcal{C}}(\omega)$ and $\tau_\alpha \in HT_{\mathcal{C}}(\omega \to X)$.*

From that we will be able to define the represented function.

4.2 Representants of integers

The following result is useful in the third part of the proof. Indeed, instead of taking an arbitrary τ_m, since we are in a context of single threaded strategies, we can directly use τ_m:

Proposition 6. *If τ is a single threaded total strategy on ω, then there is a unique $m \in \omega$ such that $\tau = \tau_m$.*

Proof. Indeed, since q_ω^O is a legal play and τ is total, then there exists a unique m such that $q_\omega^O m^P \in \tau$. Then, since τ is single threaded, we get that $\tau = (q_\omega^O m^P)^* = \tau_m$. ☐

4.3 Describing the plays of τ_α

In the second part of the proof, we need to describe an interaction between σ, τ_α, and τ_m. For that, we first give a description of the well bracketed plays of τ_α:

Proposition 7. *If $\alpha \in X^\omega$, the set of well bracketed plays of τ_α are the even-length prefixes of the words of the language $\mathcal{L}_{\mathbf{P}}$ generated by the grammar:*

$$\mathbf{P} \ ::= \ \epsilon \ \mid \ \mathbf{P} \ q_X^O \ q_\omega^P \ \mathbf{P} \ i^O \ \alpha_i^P$$

Proof. We have $\epsilon \in \tau_\alpha$, and if $u_1, u_2 \in \mathcal{L}_{\mathbf{P}}$, then:

$$\ulcorner u_1 q_X^O q_\omega^P u_2 i^O \alpha_i^P \urcorner = q_X^O q_\omega^P i^O \alpha_i^P \in \tau_\alpha$$

For the converse implication we first show that if $u \in \tau_\alpha$ has no pending question, then $u \in \mathcal{L}_{\mathbf{P}}$, by induction on the size of u:

- $u = \epsilon$: it is immediate
- $|u| > 0$: $u \in \tau_\alpha$, so $|u|$ is even, and u has no pending question, so u ends with an answer of player, which is necessarily some α_i^P by definition of τ_α. Again by definition of τ_α, the preceding move is i^O, which is the answer of a q_ω^P appearing before. But the only possible move before this q_ω^P is q_X^O, by definition of τ_α. Thus u can be written as:

  $$u = \ u_0 \ q_X^O \ q_\omega^P \ u_1 \ i^O \ \alpha_i^P$$

  By well bracketing, we know that this α_i^P is the answer to this q_X^O. By closure of τ_α under even-length prefixes, we know that $u_0 \in \tau_\alpha$. Moreover, since u has no pending question and this α_i^P is the answer to this q_X^O, we know that u_0 has no pending question, so by induction hypothesis, $u_0 \in \mathcal{L}_{\mathbf{P}}$. u_1 has no pointer to $u_0 q_X^O q_\omega^P$: indeed, an answer in u_1 points to the last pending question, but u_0 has no pending question and q_X^O and q_ω^P are answered after u_1, and by definition of τ_α, each q_ω^P points to the preceding move, which is a q_X^O. Therefore we have:

$$\ulcorner u_0 q_X^O q_\omega^P u_1 \urcorner = \ulcorner u_1 \urcorner$$

but $u_0 q_X^O q_\omega^P u_1 \in \tau_\alpha$, so $\ulcorner u_1 \urcorner \in \tau_\alpha$, and so $u_1 \in \tau_\alpha$. Now, since the last pending question of $u_0 q_X^O q_\omega^P u_1$ is q_ω^P, we know that u_1 has no pending question, and so by induction hypothesis, $u_1 \in \mathcal{L}_{\mathbf{P}}$. Finally we can conclude that

$$u = \; u_0 \;\; q_X^O \;\; q_\omega^P \;\; u_1 \;\; i^O \;\; \alpha_i^P \;\; \in \mathcal{L}_{\mathbf{P}}$$

Now we show that each $u \in \tau_\alpha$ can be extended to a $uv \in \tau_\alpha$ which has no pending question, by induction on the number of pending questions of u:

- u has no pending question: it is immediate
- u has at least one pending question: observe that the last pending question of u is a q_ω^P: indeed, each q_X^O is immediately followed by a q_ω^P, and each answer to a q_ω^P by O is immediately followed by an answer to the corresponding q_X^O by P. Since each q_ω^P is immediately preceded by the corresponding q_X^O, we can write:
$$u = u_0 q_X^O q_\omega^P u_1$$
where this q_ω^P is the last pending question. But then $u_0 q_X^O q_\omega^P u_1 0^O \alpha_0^P \in \tau_\alpha$ has strictly less pending questions than u, so we can conclude by induction hypothesis.

We deduce from the two preceding facts that each play of τ_α is an even-length prefix of a word of $\mathcal{L}_{\mathbf{P}}$. ☐

Now we can apply it to the description of the interactions between σ, τ_α and τ_m:

Proposition 8. *Let $m \in \omega, \alpha \in X^\omega$, and let $u = q_Y^O v y^P$ for some $y \in Y$ be a well bracketed play on the arena $(\omega \to X) \to (\omega \to Y)$ which satisfies $u_{|\omega \to X} \in \tau_\alpha$, $u_{|\omega} \in \tau_m$ and $u_{|Y} = q_Y^O y^P$. Then v is in the language $\mathcal{L}_{\mathbf{Q}}$ generated by the following grammar:*

$$\mathbf{Q} \;::=\; \epsilon \;\mid\; \mathbf{Q} \;\; q_X^O \;\; q_\omega^P \;\; \mathbf{Q} \;\; i^O \;\; \alpha_i^P \;\mid\; \mathbf{Q} \;\; q_\omega^P \;\; m^O$$

Proof. Since u is well bracketed and since the first move q_Y^O is answered by the last move y^P, we get that $u_{|\omega \to X} \in \tau_\alpha$ is well bracketed and has no pending question. Therefore it is in $\mathcal{L}_{\mathbf{P}}$. Since in the play P is the only one allowed to change between $\omega \to X$ and ω (in order to respect alternance), and since $\tau_m = (q_\omega^O m^P)^*$, the result follows. ☐

Remark 2. *The plays which are considered in Proposition 8 are well bracketed and obtained by an interaction between innocent strategies, so we are in a particular case of [AM96], and according to [GM00] the pointers can be erased without loss of information.*

4.4 A counter-example

We saw in the above section that well bracketing is a useful tool to dissect strategies. The fact is that it is necessary to our result:

Proposition 9. *Let $\mathcal{C}$ be the class of innocent or innocent and total strategies. Let σ be a strategy in $\mathcal{C}$ and $HT_{\mathcal{C}}((\omega \to X) \to (\omega \to Y))$ containing the following plays:*

$$\begin{array}{ccccccccc}
(\omega & \to & X) & \to & (\omega & \to & Y) \\
 & & & & & & q_Y^O \\
 & & q_X^P & & & & \\
 & & x^O & & & & \\
 & & & & & & 0^O
\end{array}
\qquad
\begin{array}{ccccccccc}
(\omega & \to & X) & \to & (\omega & \to & Y) \\
 & & & & & & q_Y^O \\
 & & q_X^P & & & & \\
q_\omega^O & & & & & & \\
 & & & & & & 1^O
\end{array}$$

Then there is no $f : X^\omega \to Y^\omega$ such that $\sigma \Vdash_{\mathcal{C}} f$.

Proof. Let τ be the single-threaded strategy on $\omega \to X$ containing the play:

$$\begin{array}{ccc}
\omega & \to & X \\
 & & q_X^O \\
 & & 0^P
\end{array}$$

Whe have:

$$\tau \Vdash_{\mathcal{C}} 0^\omega \wedge \sigma(\tau)(\tau_0) \Vdash_{\mathcal{C}} 0 \wedge \sigma(\tau_{0^\omega})(\tau_0) \Vdash_{\mathcal{C}} 1$$

$\square$

4.5 Transforming the interaction

We use the grammar above in order to define inductively, given a τ which represents α, a transformation from the interactions of σ with τ_α and τ_m to its interactions with τ and τ_m:

Definition 20. *Let $\mathcal{C}$ be any class, let $\alpha \in X^\omega$ and $\tau \Vdash_{\mathcal{C}} \alpha$. We proved in Proposition 2 that there exists a sequence of integers $(n_i)_{i \in \mathbb{N}}$ such that for all i, τ contains:*

$$\begin{array}{llcl}
\omega & & \overbrace{q_\omega^P \; i^O \; \cdots \; q_\omega^P \; i^O}^{n_i \times} & \\
\downarrow & & & \\
X & q_X^O & & \alpha_i^P
\end{array}$$

For $u = q_Y^O v y^P$ with $y \in Y$ a well bracketed play in the arena $(\omega \to X) \to (\omega \to Y)$ such that $u_{|\omega \to X} \in \tau_\alpha$, $u_{|\omega} \in \tau_m$ and $u_{|Y} = q_Y^O y^P$, we define by induction $\varphi_\tau(v)$ by:

$$\begin{aligned}
\varphi_\tau(\epsilon) &= \epsilon \\
\varphi_\tau(v_0 q_\omega^P m^O) &= \varphi_\tau(v_0) q_\omega^P m^O \\
\varphi_\tau(v_0 q_X^P q_\omega^O v_1 i^P \alpha_i^O) &= \varphi_\tau(v_0) q_X^P (q_\omega^O \varphi_\tau(v_1) i^P)^{n_i} \alpha_i^O
\end{aligned}$$

The following result proves that if τ is innocent, then by applying the above transformation we obtain an interaction with τ:

Proposition 10. *Let $u = q_Y^O v y^P$ for some $y \in Y$ be a well bracketed play in the arena $(\omega \to X) \to (\omega \to Y)$ such that $u_{|\omega \to X} \in \tau_\alpha$, $u_{|\omega} \in \tau_m$ and $u_{|Y} = q_Y^O y^P$. For any $\mathcal{C}$, if $\alpha \in X^\omega$ and $\tau \Vdash_{\mathcal{C}} \alpha$ is innocent, then $u' = q_Y^O \varphi_\tau(v) y^P$ is such that $u'_{|\omega \to X} \in \tau$, $u'_{|\omega} \in \tau_m$ and $u'_{|Y} = q_Y^O y^P$.*

Proof. First by definition of u' and φ_τ, we have $u'_{|\omega} \in \tau_m$ and $u'_{|Y} = q^O_Y y^P$. It is also straightforward to verify that $\varphi_\tau(v)_{|\omega\to X} = \varphi_\tau(v_{|\omega\to X})$. Then we only need to prove that if $v \in \tau_\alpha$, then $\varphi_\tau(v) \in \tau$. We prove this by induction on the derivation of v:

- $v = \epsilon$: $\varphi_\tau(v) = \epsilon \in \tau$
- $v = v_0 q^O_X q^P_\omega v_1 i^O \alpha^P_i$: we prove that every even-length prefix w of $\varphi_\tau(v)$ is such that $\ulcorner w \urcorner \in \tau$:
  - if w is a prefix of $\varphi_\tau(v_0)$, since by induction hypothesis $\varphi_\tau(v_0) \in \tau$ we get $w \in \tau$
  - if $w = \varphi_\tau(v_0) q^O_X (q^P_\omega \varphi_\tau(v_1) i^O)^k q^P_\omega$, then $\ulcorner w \urcorner = q^O_X (q^P_\omega i^O)^k q^P_\omega \in \tau$
  - if $w = \varphi_\tau(v_0) q^O_X (q^P_\omega \varphi_\tau(v_1) i^O)^k q^P_\omega w'$ for some non empty prefix w' of $\varphi_\tau(v_1)$, then since no move in w' point to a move in $\varphi_\tau(v_0) q^O_X (q^P_\omega \varphi_\tau(v_1) i^O)^k q^P_\omega$ we get $\ulcorner w \urcorner = \ulcorner w' \urcorner$, so $w \in \tau$ because w' is an even-length prefix of $\varphi_\tau(v_1)$ which is in τ by induction hypothesis, and finally $\ulcorner \varphi_\tau(v) \urcorner = q^O_X (q^P_\omega i^O)^{n_i} \alpha^P_i \in \tau$.

□

4.6 The theorem

We now have all the necessary material to prove the last theorem:

Theorem 3. *Let $\mathcal{C}$ be a class which contains only innocent strategies. If σ is in $\mathcal{C}$ and if it is well bracketed and hereditarily total on $X^\omega \to Y^\omega$, then there exists $f : X^\omega \to Y^\omega$ such that $\sigma \Vdash_{\mathcal{C}} f$.*

Proof. Let first define f. If $\alpha \in X^\omega$ and $m \in \omega$, then $\sigma(\tau_\alpha)(\tau_m)$ is in $HT_{\mathcal{C}}(Y)$ by Proposition 5, so it is total, so there is a unique $y \in Y$ such that $q^O_Y y^P \in \sigma(\tau_\alpha)(\tau_m)$, so $\sigma(\tau_\alpha)(\tau_m) \Vdash_{\mathcal{C}} y$. Let then define $f(\alpha)_m = y$. Now, since the only single threaded representant of $m \in \omega$ is τ_m by Proposition 6, we must show:

$$\forall \alpha, \forall \tau \Vdash_{\mathcal{C}} \alpha, \forall m, \sigma(\tau)(\tau_m) \Vdash_{\mathcal{C}} f(\alpha)_m$$

Let $\alpha \in X^\omega$ and $m \in \omega$. We have $q^O_Y f(\alpha)^P_m \in \sigma(\tau_\alpha)(\tau_m)$ so:

$$\exists u = q^O_Y v f(\alpha)_m \in \sigma, u_{|\omega\to X} \in \tau_\alpha \wedge u_{|\omega} \in \tau_m \wedge u_{|Y} = q^O_Y f(\alpha)^P_m$$

Now, let $\tau \Vdash_{\mathcal{C}} \alpha$. We have that $u' = q^O_Y \varphi_\tau(v) f(\alpha)^P_m$ is such that $u'_{|\omega\to X} \in \tau$, $u'_{|\omega} \in \tau_m$ and $u'_{|Y} = q^O_Y f(\alpha)^P_m$ by Proposition 10, so we only have to show that $u' \in \sigma$ in order to prove that $\sigma \Vdash_{\mathcal{C}} f$. Let define $\tilde{\sigma}$ as the set of legal plays which have all their even-length prefixes in σ. In other words it is the set of plays of σ plus the set of plays of σ followed by a legal O-move. We first prove that if $w_1 \in \mathcal{L}_{\mathbf{Q}}$, then $\ulcorner q^O_Y w_0 \varphi_\tau(w_1) \urcorner = \ulcorner q^O_Y w_0 w_1 \urcorner$ by induction on the production of w_1:

- $w_1 = \epsilon$: $\ulcorner q^O_Y w_0 \urcorner = \ulcorner q^O_Y w_0 \urcorner$
- $w_1 = v_0 q^P_\omega m^O$:

$$\begin{aligned}
\ulcorner q^O_Y w_0 \varphi_\tau(v_0) q^P_\omega m^O \urcorner &= \ulcorner q^O_Y w_0 \varphi_\tau(v_0) \urcorner q^P_\omega m^O \\
&= \ulcorner q^O_Y w_0 v_0 \urcorner q^P_\omega m^O \\
&= \ulcorner q^O_Y w_0 v_0 q^P_\omega m^O \urcorner
\end{aligned}$$

- $w_1 = v_0 q^P_X q^O_\omega v_1 i^P \alpha^O_i$:

$$\begin{aligned}
\ulcorner q^O_Y w_0 \varphi_\tau(v_0) q^P_X (q^O_\omega \varphi_\tau(v_1) i^P)^{n_i} \alpha^O_i \urcorner &= \ulcorner q^O_Y w_0 \varphi_\tau(v_0) \urcorner q^P_X \alpha^O_i \\
&= \ulcorner q^O_Y w_0 v_0 \urcorner q^P_X \alpha^O_i \\
&= \ulcorner q^O_Y w_0 v_0 q^P_X q^O_\omega v_1 i^P \alpha^O_i \urcorner
\end{aligned}$$

Now, we prove the property:

$$P(w_1) \quad := \quad \forall w_0, \quad (q_Y^O w_0 w_1 \in \tilde{\sigma} \ \wedge \ w_1 \in \mathcal{L}_{\mathbf{Q}}) \quad \Rightarrow \quad q_Y^O w_0 \varphi_\tau(w_1) \in \tilde{\sigma}$$

We reason by induction on the production of w_1 by $\mathbf{Q}$:

- $w_1 = \epsilon$: if $q_Y^O w_0 \in \tilde{\sigma}$, then $q_Y^O w_0 \in \tilde{\sigma}$
- $w_1 = v_0 q_\omega^P m^O$: we have $q_Y^O w_0 v_0 q_\omega^P \in \sigma$ and $\ulcorner q_Y^O w_0 \varphi_\tau(v_0) \urcorner = \ulcorner q_Y^O w_0 v_0 \urcorner$, so by innocence of σ, $q_Y^O w_0 \varphi_\tau(v_0) q_\omega^P \in \sigma$, and so $q_Y^O w_0 \varphi_\tau(v_0) q_\omega^P m^O \in \tilde{\sigma}$
- $w_1 = v_0 q_X^P q_\omega^O v_1 i^P \alpha_i^O$:
$$q_Y^O w_0 \varphi_\tau(w_1) = q_Y^O w_0 \varphi_\tau(v_0) q_X^P (q_\omega^O \varphi_\tau(v_1) i^P)^{n_i} \alpha_i^O$$
  $q_Y^O w_0 v_0$ is in $\tilde{\sigma}$ as a prefix of $q_Y^O w_0 w_1 \in \tilde{\sigma}$, so by induction hypothesis, $q_Y^O w_0 \varphi_\tau(v_0) \in \tilde{\sigma}$. Then $\ulcorner q_Y^O w_0 \varphi_\tau(v_0) \urcorner = \ulcorner q_Y^O w_0 v_0 \urcorner$ and $q_Y^O w_0 v_0 q_X^P \in \sigma$, so $q_Y^O w_0 \varphi_\tau(v_0) q_X^P \in \sigma$. Let show by induction on $0 \leq k \leq n_i$ that:
$$q_Y^O w_0 \varphi_\tau(v_0) q_X^P (q_\omega^O \varphi_\tau(v_1) i^P)^k \in \sigma$$
  For $k = 0$, we just proved it. Let now suppose this holds for some $k < n_i$. For any v_1' prefix of v_1, we have:
$$\begin{aligned}\ulcorner q_Y^O w_0 \varphi_\tau(v_0) q_X^P (q_\omega^O \varphi_\tau(v_1) i^P)^k q_\omega^O v_1' \urcorner &= \ulcorner q_Y^O w_0 \varphi_\tau(v_0) \urcorner q_X^P q_\omega^O v_1' \\ &= \ulcorner q_Y^O w_0 v_0 \urcorner q_X^P q_\omega^O v_1' \\ &= \ulcorner q_Y^O w_0 v_0 q_X^P q_\omega^O v_1' \urcorner\end{aligned}$$
  because the O-moves of v_1' point in v_1', so by innocence of σ and by induction on $|v_1'|$, we have
$$q_Y^O w_0 \varphi_\tau(v_0) q_X^P (q_\omega^O \varphi_\tau(v_1) i^P)^k q_\omega^O v_1' \in \tilde{\sigma}$$
  Then we have
$$q_Y^O w_0 \varphi_\tau(v_0) q_X^P (q_\omega^O \varphi_\tau(v_1) i^P)^k q_\omega^O v_1 \in \tilde{\sigma}$$
  and by induction hypothesis:
$$q_Y^O w_0 \varphi_\tau(v_0) q_X^P (q_\omega^O \varphi_\tau(v_1) i^P)^k q_\omega^O \varphi_\tau(v_1) \in \tilde{\sigma}$$
  But:
$$\ulcorner q_Y^O w_0 \varphi_\tau(v_0) q_X^P (q_\omega^O \varphi_\tau(v_1) i^P)^k q_\omega^O \varphi_\tau(v_1) \urcorner = \ulcorner q_Y^O w_0 v_0 q_X^P q_\omega^O v_1 \urcorner$$
  and $q_Y^O w_0 v_0 q_X^P q_\omega^O v_1 i^P \in \sigma$, so $q_Y^O w_0 \varphi_\tau(v_0) q_X^P (q_\omega^O \varphi_\tau(v_1) i^P)^{k+1} \in \sigma$ by innocence of σ. Finally we get: $q_Y^O w_0 \varphi_\tau(v_0) q_X^P (q_\omega^O \varphi_\tau(v_1) i^P)^{n_i} \alpha_i^O \in \tilde{\sigma}$

We can now prove that $u' \in \sigma$. Indeed, by applying the preceding property with $w_0 = \epsilon$ and $w_1 = v$, since $q_Y^O v \in \tilde{\sigma}$, we get $q_Y^O \varphi_\tau(v) \in \tilde{\sigma}$. Moreover, since $\ulcorner q_Y^O \varphi_\tau(v) \urcorner = \ulcorner q_Y^O v \urcorner$ and $u = q_Y^O v f(\alpha)_m \in \sigma$, we conclude by innocence of σ that $u' \in \sigma$, which achieves the proof. □

5 Conclusion

In this work, we defined a notion of representation of functions on infinite sequences by strategies of Hylan-Ong games. We then proved that in the context of single threaded strategies, the continuous functions are exactly those which are representable. Finally we obtained a result of completeness which says that innocent well bracketed hereditarily total strategies represent continuous functions. As expected, our notion of representation works better with innocent strategies. However, Theorem 2 shows that continuous functions are representable w.r.t. not necesserarily innocent strategies.

The next step of this work is to investigate the notion of hereditary totality more closely.

Further work could be to investigate realizability models directly based on games semantics, and maybe obtain models of realizability by programs with refences. Realizability is a technique to extract a program from a formal proof of a given formula. If the formula is of the form $\forall x \exists y A[x, y]$, then the extracted program should represent a function which, given x, provides y such that $A[x, y]$ is true. In the framework of functions from infinite sequences to infinite sequences, the programs extracted from intuitionistic proofs compute continuous functions, but in a classical setting, some non continuous functions can be obtained. However some of them belong to particular classes of functions (between continuous and borel) which can be modelized by variations on Wadge games (see [Sem09]). This work shows that the representation of sequences by arrow usual arenas is not suitable for an adaptation of the techniques described in [Sem09].

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